Radiationless Internal Vibrations of Extended Charged Particles in a Constant Magnetic Field

W. Heudorfer and M. Sorg

Institut für Theoretische Physik der Universität Stuttgart

The extended particle theory recently proposed is further investigated. Especially, it is shown for the force-free case that no runaway solutions exist and that the particle must ultimately assume constant velocity. For a certain structure of the particle, internal radiationless vibrations are found and discussed extensively for the case, where the particle moves in a constant homogenious magnetic field. If the magnetic field strength assumes a certain discrete set of values, the motion terminates in that radiationless internal rotation; in all other cases the particle spirals inwards under emission of radiation until it comes to rest.

I. Introduction and Survey of Results

In two recent papers [1], Moniz and Sharp have shown within the range of validity of non-relativistic quantum electrodynamics, that the equation of motion of the radiating electron is essentially of non-local character, even in the point-particle limit. From the point of view of relativistic quantum electrodynamics, this seems to be quite plausible, because the original point-like particle creates electron-positron pairs in its immediate vicinity (vacuum polarization); and since the positrons are attracted and the virtual electrons are repelled by the original point charge, there must arise a charge "atmosphere" (as Blokhintsev [2] calls it) with a certain distribution of the charge around the point particle. The linear dimensions of this polarization cloud are in the order of magnitude of the Compton wavelength $\lambda_c\,(=\hbar/2\,m\,c)$, which just enters the equation of Moniz and Sharp as a non-locality parameter.

Having in mind this concretization of the semi-classically radiating electron, one can try to find the classical equation of motion for such an extended particle, the charge distribution of which might now be described from a purely classical point of view, though its existence is originally the consequence of a quantum effect. The only difference with respect to a purely classical theory of the radiating electron would then consist in the modified dimension of the particle: the dominant length parameter is the Compton wavelength λ_c instead of the classical electron radius $r_0\,(=Z^2/2\,m\,c^2 \cong 1/137\,\lambda_c)$. However, since the numerical value of this length parameter (called Δs hereafter) is itself not derivable from

Reprint requests to M. Sorg, Institut für Theoretische Physik der Universität, Pfaffenwaldring 57, *D-7000 Stuttgart 80*.

such a classical model, it must not be specified a priori but it can be fitted later in order to meet with experimental data or/and with a higher level theory. Such a higher level theory might be relativistic quantum electrodynamics or (e. g.) the chronon theory of Caldirola [3, 4].

In a recent paper [5], we have presented a covariant framework for arbitrarily extended, classical, charged particles being characterized by certain time-like structure functions. In the present paper, we want to study some specific effects occurring in connection with such extended particles: these are the internal self-oscillations, which are excited and sustained by external forces of appropriate frequency.

If one considers namely the elementary length interval Δs as a characteristic time parameter of the particle, one would expect internal damped oscillations of frequency Δs^{-1}, which is the inverse propagation time of a light signal running through the particle; and whenever the external force changes appreciably over proper time intervals of magnitude Δs, those internal oscillations could be excited in principle. The result would be a sort of "resonance" between the internal modes and the external force such that certain of the modes become stationary and radiationless, and the particle would be left in a stationary excited state in the given force field.

It is exactly this phenomenon, which we intend to demonstrate explicitly in this paper by means of the recently proposed non-local equation of motion for the classically radiating electron [6]

$$m_{\text{mech}}\,c^2\,\dot{u}^\lambda + m_{\text{el}}\,c^2\left\{\frac{\Delta u^\lambda}{\Delta s} - \left(u\cdot\frac{\Delta u}{\Delta s}\right)u^\lambda\right\} = K^\lambda. \quad (\text{I, 1})$$

In contrast to the previously studied one-dimensional case [7], we turn now to a two-dimensional

problem and choose for the external force a constant uniform magnetic field

$$K^\lambda = Z\, F^{\mu\lambda}\, u_\mu \qquad (I, 2)$$

with

$$-F^{21} = F^{12} = B = \text{const} \qquad (I, 3)$$

and all other components of the field strength tensor vanishing.

Under these conditions, we find that for the discrete set of magnetic field strengths

$$B_n = |Z|^{-1}\, m_{\text{mech}}\, c^2\, \omega_n$$

with the cyclotron frequencies $\omega_n = n \cdot 2\,\pi/\Delta s$ there exist circular radiationless motions, where the radius of the circles is dependent upon the initial condition but never exceeds the extension parameter Δs. Thus, it is clear, that those motions are internal rotations of the extended particle.

These results also apply to the non-relativistic limit, where the spectral decomposition of an arbitrary motion with respect to the internal decaying modes is easily feasible by means of the Laplace transform method. One can demonstrate here explicitly, how the various damping constants of the internal modes decrease (with increasing magnetic field strength B) down to zero for the special values B_n from above. We present numerical results of how the stationary circles are approached in the course of the motion starting with a constant impact velocity as initial condition.

The longitudinal motion (parallel to the magnetic lines of force) is also studied, and it is found that this longitudinal motion is rapidly damped down to constant velocity, as well for arbitrary values of B, where the particle spirals inwards until it comes to rest in the transverse plane as also for the above "resonant" values B_n of the magnetic field, where the final state of motion is such an internal rotation. Hence, one might say from a macroscopic point of view that the longitudinal velocity is "approximately" a constant of the motion.

II. Existence of Radiationless Motions

First, we want to show that the equation of motion (I, 1) is the only one of the general class [5]

$$m_{\text{mech}}\, c^2\, \dot{u}^\lambda + m_{\text{el}}\, c^2 \{\dot{Q}^\lambda - (\dot{Q}\,u)\,u^\lambda\} = K^\lambda \qquad (II, 1)$$

where

$$Q^\lambda = \int_0^{\Delta s} f_{(\sigma)}\, u^0_{(s-\sigma)}\, \mathrm{d}\sigma/\Delta s\,, \qquad (II, 2)$$

which admits stationary radiationless motions for a force K^λ, which itself does not transfer energy to the particle [e. g. magnetic force $(I; 2, 3)$]. To this end, assume that there be a periodic motion of proper time period $\overline{\Delta s}$

$$u^\lambda_{(s)} = u^\lambda_{(s-\overline{\Delta s})}\,, \quad \forall s \qquad (II, 3)$$

so that an integration of $(II, 1)$ yields on account of

$$\int_s^{s+\overline{\Delta s}} K^0_{(s')}\, \mathrm{d}s' = 0\,, \quad \forall s \qquad (II, 4)$$

the following result $(z_0 \equiv c\,t;\ \mathrm{d}z^\lambda/\mathrm{d}s \equiv u^\lambda)$

$$m_{\text{mech}}\, c^2\, [u^0]_s^{s+\overline{\Delta s}} + m_{\text{el}} \int_0^{\Delta s} \frac{\mathrm{d}\sigma}{\Delta s}\, f_{(\sigma)}\, [u^0_{(s-\sigma)}]_s^{s+\overline{\Delta s}}$$

$$= m_{\text{el}}\, c^3 \int_{t_{(s)}}^{t(s+\overline{\Delta s})} (\dot{Q}\,u)\, \mathrm{d}t\,. \qquad (II, 5)$$

Because of the periodicity condition $(II, 3)$, the first two integrals on the left of $(II, 5)$ vanish and we are led to the requirement

$$\int_{t_{(s)}}^{t_{(s+\overline{\Delta s})}} (\dot{Q}\,u)\, \mathrm{d}t = 0\,, \quad \forall s\,. \qquad (II, 6)$$

But the invariant radiation rate $\Re$

$$\Re = - m_{\text{el}}\, c\, (\dot{Q}\,u)$$

$$= m_{\text{el}}\, c\, \{\Delta s^{-1}\, f_{(\Delta s)}\, [u^\lambda_{(s)}\, u_{\lambda(s-\Delta s)} - 1]$$

$$- \int_0^{\Delta s} f'_{(\sigma)}\, [u^\lambda_{(s)}\, u_{\lambda(s-\sigma)} - 1]\, \mathrm{d}\sigma/\Delta s\} \qquad (II, 7)$$

is positive-semidefinit [5] on account of the restrictions imposed on the structure function $f_{(\sigma)}$:

$$f_{(\Delta s)} \geq 0\,; \quad f'_{(\sigma)} \leq 0 \qquad (II, 8)$$

and hence we must have $\Re \equiv 0$ along the trajectory. However, inspection of $(II, 7)$ shows that the radiation rate can vanish only in two cases:

$\alpha)\quad f'_{(\sigma)} \neq 0$ for $\sigma \in [\sigma_1, \sigma_2] \subseteq [0, \Delta s]\,; \quad \sigma_2 > \sigma_1$.

$$\qquad (II, 9)$$

In this case, the radiation rate $\Re$ vanishes only if $u^\lambda = \text{const.}(s)$ and the only possible stationary motion in the force field under consideration is uniform motion [e. g. parallel to the magnetic lines of force in example $(I, 3)$].

$\beta)\quad f'_{(\sigma)} \equiv 0\,; \quad f_{(\sigma)} \equiv 1\,; \quad u^\lambda(s - \Delta s/n) = u^\lambda_{(s)}$

$$n = 1, 2, 3, \ldots\,. \qquad (II, 10)$$

For this special choice of $f_{(\sigma)}$, the general equation of motion $(II, 1)$ reduces to $(I, 1)$; and we see that the proper time period Δs must be equal to the

fundamental time interval $\Delta s/n$. For this reason, the electromagnetic fourmomentum P_{el}^μ

$$P_{el}^\mu := m_{el} c Q^\mu = m_{el} c \Delta s^{-1} (z^\mu_{(s)} - z^\mu_{(s-\Delta s)}) \quad (II, 10)$$

vanishes also ($P_{el}^\mu \equiv 0, \forall s$) and Eq. (I, 1) assumes the well-known, non-radiative form

$$m_{mech} c^2 \dot{u}^\lambda = K^\lambda, \quad (II, 12)$$

which is known to admit circular motions as solutions in a constant homogeneous magnetic field (see next sections).

We see that the electromagnetic part has disappeared from the equations, and the behaviour of the particle in such a "resonant" motion is determined only by its mechanical subsystem. Moreover, our considerations have been performed completely within the framework of Special Relativity so that one does not have to fear the spuriousness of the corresponding non-relativistic linearized calculations (see Sect. 6 G of Erber's work [8]).

An obvious generalization of the structure function (II, 10), which still leads to radiationless motions, is (e. g.)

$$f'_{(\sigma)} = -\varkappa \delta_{(\sigma-\sigma_0)};$$
$$f_{(\sigma)} = \begin{cases} f_1 = \text{const}; 0 \leqq \sigma < \sigma_0 < \Delta s, \\ f_2 = \text{const}; \sigma_0 \leqq \sigma \leqq \Delta s. \end{cases} \quad (II, 13)$$

The three constants $\varkappa$, f_1, f_2 are connected by the normalization condition [5]

$$\int_0^{\Delta s} f_{(\sigma)} \frac{d\sigma}{\Delta \sigma} = 1 \quad (II, 14)$$

and the 'charge normalization' condition [5]

$$\int_0^{\Delta s} \sigma f_{(\sigma)} \frac{d\sigma}{\Delta s} = \frac{2}{3} Z^2 (m_{el} c^2)^{-1} =: \bar{\sigma} \quad (II, 15)$$

through

$$f_1 = (\Delta s + \sigma_0 - 2\bar{\sigma}) \sigma_0^{-1} \quad (II, 16a)$$

$$f_2 = (2\bar{\sigma} - \sigma_0)(\Delta s - \sigma_0)^{-1} \overset{!}{>} 0. \quad II, (16b)$$

Moreover

$$f_1 - f_2 = \varkappa = \Delta s (\Delta s - 2\bar{\sigma}) \sigma_0^{-1} (\Delta s - \sigma_0)^{-1} > 0. \quad (II, 17)$$

The radiation rate $\mathfrak{R}$ from (II, 7) then vanishes on account of the periodicty condition (assume $\sigma_0/\Delta s$ as a rational number):

$$u^\lambda_{(s)} = u^\lambda_{(s-n\sigma_0)} = u^\lambda_{(s-m\Delta s)} \, m, n \,\big|\, = 1, 2, 3, \ldots \quad (II, 18)$$

and the electromagnetic four-momentum corresponding to (II, 2) becomes

$$P_{el}^\mu = \Delta s^{-1} m_{el} c \ \{f_1[z^\mu_{(s)} - z^\mu_{(s-\sigma_0)}] \quad (II, 19)$$
$$+ f_2[z^\mu_{(s-\sigma_0)} - z^\mu_{(s-\Delta s)}]\}.$$

Obviously, such a particle structure resembles a piecewise constant "charge distribution".

Summarizing results, we have found that nontrivial radiationless stationary motions in external force fields, which do not supply energy to the particle during the stationary motion, are only possible in the case of piecewise constant structure functions $f_{(\sigma)}$ in the class of Equations (II, 1). We proceed now to analyze these stationary states in more detail.

III. Radiationless Motions in the Relativistic Case

In this section, we restrict ourselves to the motion of the particle in a plane orthogonal to the direction of the constant homogeneous B-field. The equation of motion (I, 1) in the radiationless case reduces to (II, 12) together with the periodicity condition (II, 10). Since it is already known that the only solutions of (II, 12) for the special force field (I; 2, 3) are circles, we can at once insert the ansatz

$$z^0_{(s)} = b^* s, \qquad z^2_{(s)} = A \sin(\omega s),$$
$$z^1_{(s)} = A \cos(\omega s), \qquad z^3_{(s)} = 0 \quad (III, 1)$$
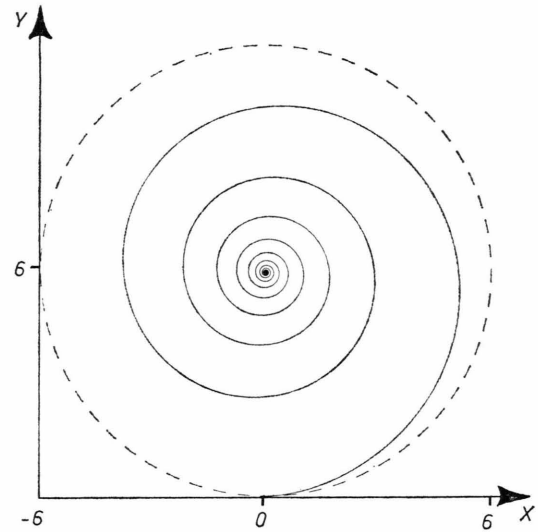

Fig. 1 Trajectory of the radiating electron according to (I, 1) in a non-resonant ($B \neq B_n$), homogeneous magnetic field. The final state of motion is the rest in the transverse plane ($x^3 = \text{const.}$). Horizontal axis: $x := x^1/\Delta s_{exp}$; vertical axis: $y := x^7/\Delta s_{exp}$; $\Delta s_{exp} = 2 Z^2/(3 m_{exp} c^2)$. The mass ratio ($m_{el}/m_{mech}$) was chosen as 0.5 and the field strength $B = 0.12 B_1$. Initial condition in $[0, \Delta s]$: $u^3 = u^2 = 0$; $u^1 = u = 0.5$. The dashed circle is the non-radiating particle trajectory following from (II, 12) with m_{mech} replaced by m_{exp}.

with

$$u^\lambda u_\lambda = 1 \Rightarrow b^* = + \sqrt{1 + (A\,\omega)^2} \quad \text{(III, 2)}$$

into the reduced set of equations

$$m_{\text{mech}}\, c^2\, \dot{u}^1 = Z\,B\,u^2\,,$$
$$m_{\text{mech}}\, c^2\, \dot{u}^2 = -\,Z\,B\,u^1\,, \quad \text{(III, 3)}$$

to obtain

$$m_{\text{mech}}\, c^2\, \omega = -\,Z\,B = |Z| \cdot B\,. \quad \text{(III, 4)}$$

The cyclotron frequency ω, if measured in proper time units, is fixed by the strength of the B-field alone, independent from the energy of the particle and the initial conditions.

But now we have to add the periodicity condition $u^\lambda{}_{(s)} = u^\lambda{}_{(s-\Delta s)}$, which selects the frequency ω as

$$\omega_n\, \Delta s = n \cdot 2\,\pi\,, \quad n = 1, 2, 3, \ldots \quad \text{(III, 5)}$$

and this, conversely, fixes the B-field as

$$B_n = |Z|^{-1}\, m_{\text{mech}}\, c^2\, \omega_n\,. \quad \text{(III, 6)}$$

Only if the magnetic field strength coincides with one of these discrete values B_n, radiationless circular trajectories are possible. In all other cases, where $B \neq B_n$, the particle spirals inwards as expected under emission of radiation until it comes to rest in
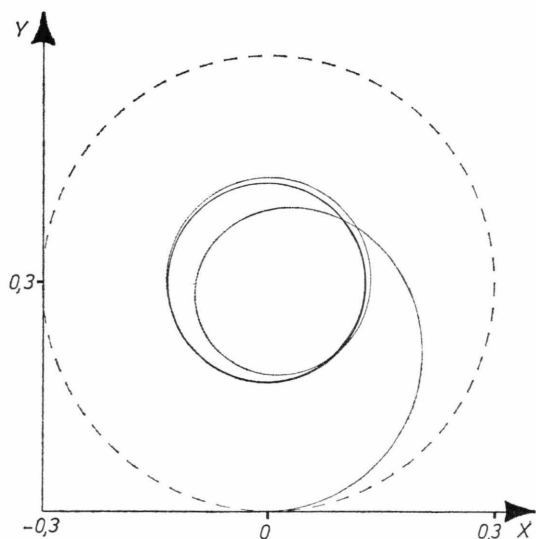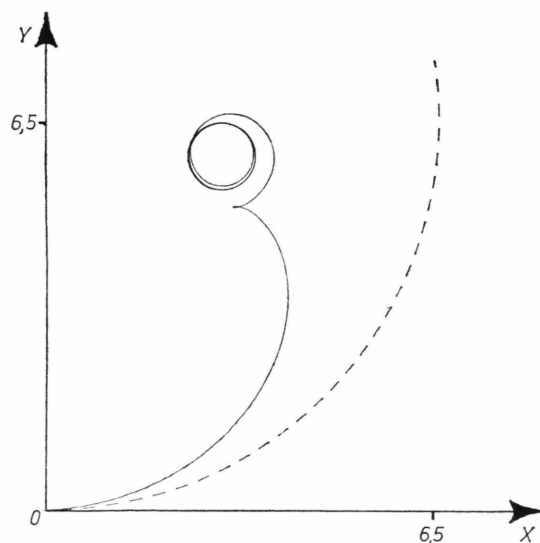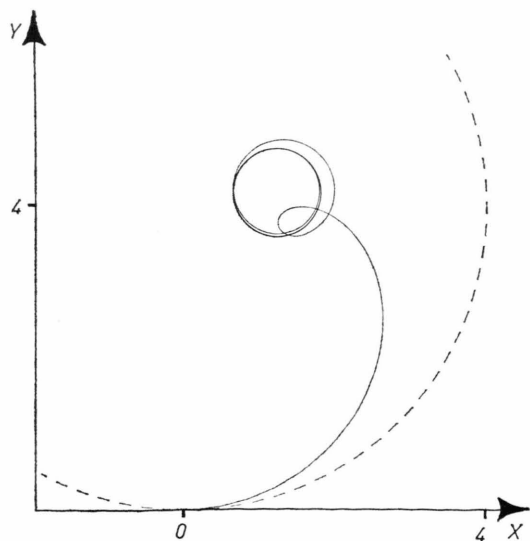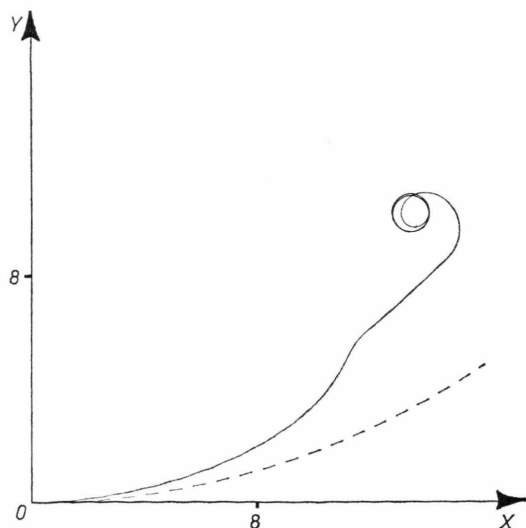

Fig. 2.

Fig. 4.

Fig. 3.

Fig. 5.

Fig. 2—5. Particle trajectory in the resonant case $B = B_1$. Parameters are unchanged with respect to Figure 1. The initial velocity u is $u = 0.2$ (Fig. 2); $u = 2.8$ (Fig. 3); $u = 4.5$ (Fig. 4) and $u = 20$ (Figure 5).

a certain point dependent from the initial conditions. Figure 1 shows a numerical solution of this sort.

It does, however, not seem to be worthwhile studying the difference in energy loss if compared with the corresponding results of the Lorentz-Dirac theory or other theories of radiation reaction (e. g. Mo-Papas [9]). All these equations reduce in lowest order to the form [10]

$$m \, c^2 \, \dot{u}^\mu = K^\mu + \tfrac{8}{3} \, r_0{}^2 \left\{ - F^{\mu\lambda} \, F^\nu{}_\lambda \, u_\nu + (F^{\sigma\lambda} \, F^\nu{}_\lambda \, u_\nu \, u_\sigma) u^\mu \right\}$$

which was recently proposed as an exact equation [11]. The classical corrections for the energy loss following from this equation are extremely small and commonly masked by quantum effects [9, 12, 13].

On the other hand, the resonant field strengths are extremely high and hardly attainable in a terrestrial experiment. For a rough evaluation we can put

$$B_1 \approx |Z|^{-1} \, r_0{}^{-1} \, 2 \, \pi \, m_{\mathrm{exp}} \, c^2 \qquad \text{(III, 7)}$$

and one recognizes this as the same magnitude as an electric field strength would have "on the surface" of a spherical electron with classical electron radius r_0. But it seems plausible that the external field strength must come in the order of magnitude of the field strengths governing the interior of the electron, so as to produce a "resonance" with the internal modes.

For a numerical demonstration of how the particle approaches the stationary final state, we have chosen $n = 1$ ($\Rightarrow$ 1 circular revolution per proper time interval Δs) and we have shot in the particle with constant velocity u $[\equiv |\boldsymbol{v}/c| \cdot (1 - \boldsymbol{v}^2/c^2)^{-1/2}]$ as initial condition in the first Δs-interval. Figures 2 to 5 show a plot of the trajectories for a wide range of initial velocities u. The dashed circle would be followed by a non-radiating particle with same experimental mass m_{exp} and same initial condition, according to Eq. (II, 12) with m_{mech} replaced by m_{exp} ($= m_{\mathrm{mech}} + m_{\mathrm{el}}$).

From these figures, one recognizes that the radius (A) of the final circle depends strongly from the initial condition u. The magnitude of the final radius is indeed a crucial test to our interpretation that the final radiationless motion be a sort of "internal resonance" of the extended particle. In order that this interpretation be true, the final radius (A) should not exceed the size (Δs) of the particle so that the motion can be viewn as an internal rotation. Figure 6 shows a plot of numerically calculated

final radii as function of the initial velocity u; and one recognizes indeed: $A < \Delta s$, even for very high impact velocities ($u = 10^3 \approx u^0{}_{\mathrm{in}} = \mathfrak{E}/m_{\mathrm{exp}} \, c^2$).

Next, one would like to know the energy remaining in the circular motion. Obviously, the whole initial energy $\mathfrak{E}_{\mathrm{in}} = m_{\mathrm{exp}} \, c^2 \, u^0{}_{\mathrm{in}}$ cannot be radiated away because of that final energy connected with the circular motion. The latter part of the energy is [observe the first of (III, 1) and (III, 2)]

$$\mathfrak{E}_{\mathrm{fin}} = m_{\mathrm{mech}} \, c^2 \, u^0_{\mathrm{fin}} = m_{\mathrm{mech}} \, c^2 \, b^*$$
$$= m_{\mathrm{mech}} \, c^2 \, \sqrt{1 + (2 \, \pi \, n \, A \, \Delta s^{-1})^2} \qquad \text{(III, 8)}$$

so that the fraction (e_{rad}) of the radiated energy becomes

$$e_{\mathrm{rad}} = \mathfrak{E}_{\mathrm{in}}^{-1} (\mathfrak{E}_{\mathrm{in}} - \mathfrak{E}_{\mathrm{fin}}) = (u^0_{\mathrm{in}})^{-1} \, (u^0_{\mathrm{in}}$$
$$- \frac{m_{\mathrm{mech}}}{m_{\mathrm{exp}}} \sqrt{1 + (2 \, \pi \, n \, A \, \Delta s^{-1})^2} . \quad \text{(III, 9)}$$

This quantity is plotted for $n = 1$ in Figure 7, and one recognizes that for low impact velocities ($u \to 0$) the final energy equals roughly the constant fraction $1 - m_{\mathrm{mech}}/m_{\mathrm{exp}} = m_{\mathrm{el}}/m_{\mathrm{exp}}$, whereas for high impact velocities the most part of the initial energy is radiated away ($e_{\mathrm{rad}} \to 1$). So we see that, in the non-relativistic limit, that part of the initial energy $\mathfrak{E}_{\mathrm{in}}$ ($= \mathfrak{E}_{\mathrm{mech, in}} + \mathfrak{E}_{\mathrm{el, in}}$) is radiated away, which has been carried along by the electromagnetic subsystem alone ($\mathfrak{E}_{\mathrm{el, in}} = m_{\mathrm{el}} \, c^2 \, u^0_{\mathrm{in}}$)

$$\mathfrak{E}_{\mathrm{rad}} = \mathfrak{E}_{\mathrm{el, in}} ; \quad \mathfrak{E}_{\mathrm{fin}} = \mathfrak{E}_{\mathrm{mech, in}} . \qquad \text{(III, 10)}$$
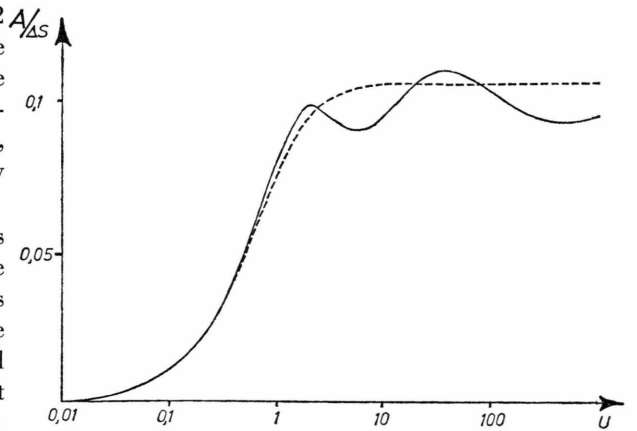

Fig. 6. Plot of the (reduced) final radius $A/\Delta s$ versus relativistic velocity u. Solid curve: numerical calculation for the relativistic case. Dashed curve: Exact result (V, 20) for the non-relativistic case.
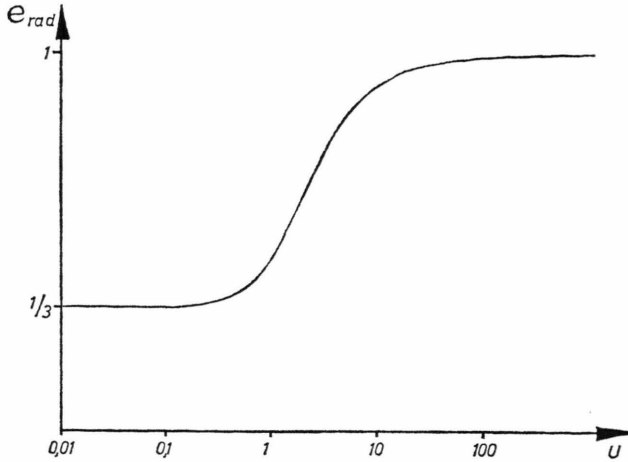
Fig. 7. The radiated part e_{rad} of the initial energy versus relativistic velocity u $(m_{el}/m_{mech}=0.5)$.

This non-relativistic result shall be verified exactly (in that limit) in the next sections.

For the sake of physical interpretation of the stationary motions, it is most instructive to consider the non-relativistic limit, and the adequate mathematical tool for this is the Laplace transform method, which we shall apply in the next section.

IV. Non-relativistic Limit, Internal Vibrations, Runaways

So far, our main results have been necessarily of numerical character, because it is difficult to solve the relativistic equations exactly. However, in the non-relativistic domain, the equations (II, 1) become linear

$$m_{mech}\frac{dv}{dt}+m_{el}\int_{\tau=0}^{\tau_0}f_{(\tau)}\frac{dv_{(t-\tau)}}{dt}\frac{d\tau}{\tau_0}=K;\ (\tau_0=c^{-1}\Delta s)$$
$$(IV,1)$$

especially (I, 1)

$$m_{mech}\frac{dv}{dt}+m_{el}\tau_0^{-1}(v_{(t)}-v_{(t-\tau_0)})=K,\qquad (IV,2)$$

and the Laplace transform method [17] can be applied to find the solution $v_{(t)}$ in closed form for any given force $K_{(t)}$.

Let us first study the force-free case $K_{(t)}\equiv 0$. Then it is assured for Eq. (IV, 2) [and presumably also for Eq (IV, 1), though this generalized form is not mentioned explicitly in Ref. [17]] that every solution $v_{(t)}$ evolving from a suitably prescribed

initial condition $\{v_{in(t)};\ t\in[0,\tau_0]\}$ is representable as an exponential series

$$v_{(t)}=\sum_{r=0}^{\infty}c_r\exp(\lambda_r t),\qquad (IV,3)$$

where the coefficients c_r are determined as follows

$$c_r=\frac{P(\lambda_r)}{h'(\lambda_r)},\qquad (IV,4)$$

the λ_r $(r=0,1,2,3,\ldots)$ being the (complex) characteristic roots of the "characteristic equation"

$$h_{(\lambda)}\equiv m_{mech}\lambda+\tau_0^{-1}m_{el}(1-\exp[-\lambda\tau_0])=0.\qquad (IV,5)$$

The initial condition enters the function $P_{(\lambda)}$ explicitly in the form

$$P_{(\lambda)}=m_{mech}v_{in(\tau_0)}\exp[-\lambda\tau_0]\qquad (IV,6)$$
$$+(m_{mech}\lambda+\tau_0^{-1}m_{el})\int_0^{\tau_0}v_{in(\tau)}\exp[-\lambda\tau]d\tau.$$

From the series representation (IV, 3) it is immediately obvious that the existence of runaway solutions depends from the existence of roots λ_r with positive real part. It was already shown in an earlier paper [5] that for Eq. (IV, 2) all characteristic roots have non-positive real parts, and the only one with vanishing real part $[Re(\lambda_0)\equiv a_0=0]$ has also vanishing imaginary part $[Im(\lambda_0)\equiv b_0=0]$. This zero root corresponds to a pure translational mode (observe that the free particle can move with arbitrary constant velocity v), whereas all others modes $(\lambda_r=a_r+i\,b_r;\ r=1,2,3,\ldots)$ are damped: $a_r<0$. Obviously, these damped modes characterize the internal vibrations of the extended particle. The damping of the modes arises from the irradiation of electromagnetic energy.

As already pointed out by Wildermuth [14], this point of view meets with a certain difficulty. Suppose that the length parameter Δs (or τ_0) tends to zero, which means that we make the size of the particle arbitrarily small. Holding the total charge (Z) fixed in this process, we see from (II, 15) that the electromagnetic mass (m_{el}) diverges to infinity, but since the experimental mass (m_{exp}) is finite and is the sum of the mechanical and electromagnetic part, the first one (m_{mech}) must become negative. This mechanism creates runaway solutions and the acausal behaviour, as was shown earlier by Wildermuth and more recently by Moniz and Sharp [1]. It is then argued that the negative value of the mechanical mass

is also the origin of the runaway solutions in the Lorentz-Dirac point approximation [15].

However, as long as we adopt $m_{\mathrm{mech}} > 0$, all those internal modes are damped, and this is true for both the special case (IV, 2) and the general class (IV, 1), as can be readily shown: Trying the ansatz (IV, 3) for (IV, 1) yields after separation into real and imaginary parts $(\lambda_r = a_r + i\,b_r)$

$$m_{\mathrm{mech}} + m_{\mathrm{el}} \int\limits_{\tau=0}^{\tau_0} f_{(\tau)} \exp[-a_r\,\tau]\,\cos(b_r\,\tau)\,\mathrm{d}\tau/\tau_0 = 0$$

$$\text{(IV, 7a)}$$

$$\int\limits_0^{\tau_0} f_{(\tau)} \exp[-a_r\,\tau]\,\sin(b_r\,\tau)\,\mathrm{d}\tau/\tau_0 = 0. \quad \text{(IV, 7b)}$$

But the second equation can only be fulfilled for non-zero b_r, if the pre-factor $(f_{(\tau)} \exp[-a_r\,\tau])$ of the sin-function is not monotonically decreasing with respect to the variable τ. Since $f_{(\tau)}$ is monotonically decreasing due to (II, 8) and (II, 9), the damping constants (a_r) must be all negative in order that (IV, 7b) can be satisfied. Hence, all modes are damped $(a_r < 0, \; \forall r)$.

The conclusion that there are no runaway solutions as long as $m_{\mathrm{mech}} > 0$, holds also for the relativistic domain. Rewriting Eq. (II, 5) as

$$m_{\exp} c^2 u^0_{(s)} + m_{\mathrm{el}} c^2 [Q^0_{(s)} - u^0_{(s)}] \quad \text{(IV, 8)}$$

$$= -c^2 \int\limits_0^{t_{(s)}} \Re\,\mathrm{d}t + m_{\exp} u^0_{(0)} + m_{\mathrm{el}} c^2 [Q^0_{(0)} - u^0_{(0)}],$$

we see that the change of kinetic energy $(m_{\exp} c^2 u^0_{(s)})$ and of internal distortion energy $(m_{\mathrm{el}} c^2 [Q^0 - u^0])$ is balanced by the emitted power $\left(c^2 \int\limits_0^t \Re\,\mathrm{d}t\right)$. But since the radiation rate $\Re$ is positive-semidefinit, the right-hand side of (IV, 8) is monotonically decreasing during the motion; however, as long as $m_{\mathrm{mech}} > 0$, the left-hand side of (IV, 8) is always positive on account of $u^0 \geqq 1$; $Q^0 \geqq 0$ $(f_{(\sigma)} \geqq 0)$. Thus, the radiated power cannot become infinite for $t \to \infty$ but must assume a finite value

$$c^2 \int\limits_0^\infty \Re\,\mathrm{d}t = \mathfrak{E}_{\mathrm{rad}} < \infty. \quad \text{(IV, 9)}$$

This is only possible, if

$$\lim_{t\to\infty} \Re_{(t)} = \lim_{t\to\infty} [-m_{\mathrm{el}}\,c\,(\dot{Q}\,u)] = 0. \quad \text{(IV, 10)}$$

and from the considerations below (II, 9) then follows

$$\lim_{t\to\infty} u^\lambda_{(s_{(t)})} = \text{const} \quad \text{(IV, 11)}$$

which completes the proof of non-existence of runaway solutions in the case $m_{\mathrm{mech}} > 0$. Obviously, this proof is valid whenever the external force does not supply energy to the particle, especially if no external force is present. Thus, the earlier proof [18] for the special case $f_{(\sigma)} \equiv 1$ and no force present is generalized now to arbitrary $f_{(\sigma)}$ from (II, 9). (Observe here:

$$\Re \equiv 0 \Rightarrow \dot{Q}^\lambda \equiv 0 \Rightarrow m_{\mathrm{mech}} c^2 \dot{u}^\lambda$$
$$= K^\lambda \equiv 0 \Rightarrow u^\lambda = \text{const.})$$

V. Non-relativistic Motion in a Constant Magnetic Field

Now we proceed to study the non-relativistic transverse motion in a constant, homogeneous B-field. Especially, we are interested here in an exact determination of the radiationless final state of motion for a given initial condition.

Starting with the general equations of motion (IV, 1)

$$m_{\mathrm{mech}} \frac{\mathrm{d}v^1(t)}{\mathrm{d}t} + m_{\mathrm{el}} \int\limits_{\tau=0}^{\tau_0} f_{(\tau)} \frac{\mathrm{d}v^1(t-\tau)}{\mathrm{d}t} \frac{\mathrm{d}\tau}{\tau_0} \quad \text{(V, 1a)}$$

$$= \frac{1}{c} Z\,B\,v^2(t),$$

$$m_{\mathrm{mech}} \frac{\mathrm{d}v^2(t)}{\mathrm{d}t} + m_{\mathrm{el}} \int\limits_{\tau=0}^{\tau_0} f_{(\tau)} \frac{\mathrm{d}v^2(t-\tau)}{\mathrm{d}t} \frac{\mathrm{d}\tau}{\tau_0}$$

$$= -\frac{1}{c} Z\,B\,v^1(t), \quad \text{(V, 1b)}$$

it is convenient to introduce the complex velocity

$$\mathfrak{v}_{(t)} = v^1(t) + i\,v^2(t) \quad \text{(V, 2)}$$

and unify the two Eqs. (V, 1) into a single one $(\varrho := m_{\mathrm{el}}/m_{\mathrm{mech}})$

$$\frac{\mathrm{d}\mathfrak{v}_{(t)}}{\mathrm{d}t} + \varrho \int\limits_0^{\tau_0} f_{(\tau)} \frac{\mathrm{d}\mathfrak{v}_{(t-\tau)}}{\mathrm{d}t} \frac{\mathrm{d}\tau}{\tau_0} = i\,\Omega\,\mathfrak{v}_{(t)}, \quad \text{(V, 3)}$$

where the cyclotron frequency Ω

$$\Omega = (m_{\mathrm{mech}}\,c)^{-1} |Z|\,B \quad (>0) \quad \text{(V, 4)}$$

has been introduced. Trying now the ansatz

$$\mathfrak{v}_{(t)} \sim \exp[\lambda\,t] \quad \text{(V, 5)}$$

yields the following characteristic equation

$$1 + \varrho \int\limits_0^{\tau_0} \frac{\mathrm{d}\tau}{\tau_0} f_{(\tau)} \exp[-\lambda\,\tau] = i\,\lambda^{-1}\,\Omega, \quad \text{(V, 6)}$$

or separating into real and imaginary part $(\lambda = a + i\,b)$

$$1 + \varrho \int_0^{\tau_0} \frac{d\tau}{\tau_0} f_{(\tau)} \exp[-a\,\tau] \cos(b\,\tau)$$
$$= (a^2 + b^2)^{-1} b\,\Omega\,, \quad \text{(V, 7a)}$$

$$\varrho \int_0^{\tau_0} \frac{d\tau}{\tau_0} f_{(\tau)} \exp[-a\,\tau] \sin(b\,\tau)$$
$$= -(a^2 + b^2)^{-1} a\,\Omega\,. \quad \text{(V, 7b)}$$

From the last equation, one realizes easily that solutions for λ with vanishing real part ($a = 0$: undamped oscillations) are only possible for the class of piecewise constant structure functions $f_{(\tau)}$, such as described under (II, 13) up to (II, 18). This non-relativistic result corresponds to the relativistic one found in Section II.

Next, we specialize formulae (V, 1) (V, 3), and (V, 7) to the simple case $f_{(\tau)} \equiv 1$ obtaining as equation of motion

$$\frac{d\mathfrak{v}_{(t)}}{dt} + \tau_0^{-1} \varrho\,(\mathfrak{v}_{(t)} - \mathfrak{v}_{(t-\tau_0)}) - i\,\Omega\,\mathfrak{v}_{(t)} = 0 \quad \text{(V, 8)}$$

with the characteristic equation

$$h(\lambda) \equiv \lambda + \tau_0^{-1} \varrho\,(1 - \exp[-\lambda\,\tau_0]) - i\,\Omega = 0 \quad \text{(V, 9)}$$

or split up into real and imaginary part ($\lambda = a + i\,b$)

$$\varrho + \tau_0 = \varrho \exp[-a\,\tau_0] \cos(b\,\tau_0) \quad \text{(V, 10a)}$$

$$-(b\,\tau_0 - \Omega\,\tau_0) = \varrho \exp[-a\,\tau_0] \sin(b\,\tau_0)\,. \quad \text{(V, 10b)}$$

From these equations it is seen easily that only $a \leq 0$ is possible. Moreover, $(a, b) = (0, 0)$ is no longer a solution as was the case for vanishing external force ($\Omega = 0$).

In order to study the stationary modes characterized by a root of the form

$$(a, b) = (0, b_n)\,; \quad n = 1, 2, 3, \ldots\,,$$

one eliminates the frequency b from the set of Eqs. (V, 10) and finds

$$\Omega\,\tau_0 = n \cdot 2\,\pi \pm \arccos\left[\frac{a\,\tau_0 + \varrho}{\varrho \exp[-a\,\tau_0]}\right] \quad \text{(V, 11)}$$

$$\pm \sqrt{1 - \left(\frac{a\,\tau_0 + \varrho}{\varrho \exp[-a\,\tau_0]}\right)^2}\;\varrho \exp[-a\,\tau_0]$$
$$n = 0, \pm 1, \pm 2, \pm 3, \ldots\,.$$

The two branches of the single curve (V, 11) have been plotted in a $(a\,\tau_0/\Omega\,\tau_0)$-diagram (Fig. 8), and one recognizes very well from this figure, that the infinite set of damping constants a_r ($r = 0, 1, 2, 3, \ldots$) for arbitrary but fixed Ω can be obtained by simply shifting a single curve (V, 11) (e. g. that for $n = 0$) parallel to the $(\Omega\,\tau_0)$-axis by the amount $n \cdot 2\,\pi$. Therefore, increasing the value of Ω from zero, the real parts of the modes (λ_r) become zero successively so that for $\Omega_n\,\tau_0 = n\,2\,\pi$ there exists always one mode with zero real part ($a_n = 0$).

Thus we arrive at an infinite set of radiationless motions with the corresponding cyclotron frequencies given by Ω_n. Observe that the intersections of two curves (V, 11) for different n do not necessarily indicate a degeneracy, because the corresponding imaginary parts b_r are in general different there.

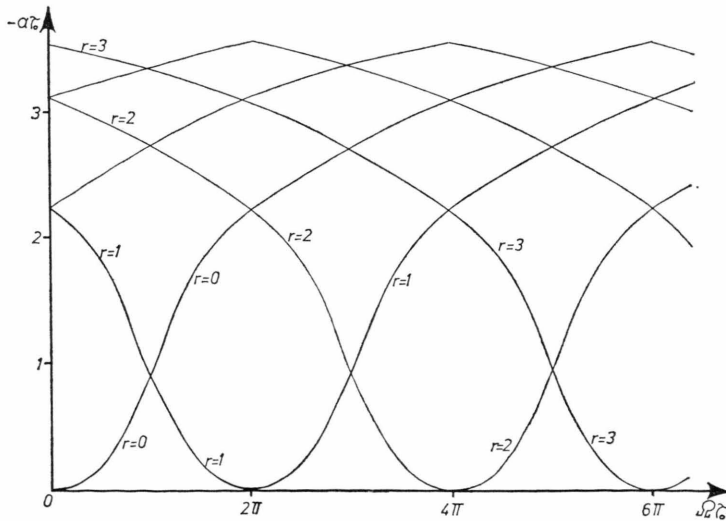Our next task is now to find the final state of motion for a given initial condition. To this end, let

Fig. 8. The first three damping constants (a_1, a_2, a_3) as function of the frequency Ω. The branches with positive ascent at $\Omega = 0$ belong to modes with negative frequency b. These modes can never come to resonance because of their opposite sense of rotation.

us first point out that the equation of motion (V, 8) has a quite similar structure to the force-free case (IV, 2); only the constant pre-factors of the (complex) velocity are different. We can therefore again apply the Laplace transform method resulting in the series representation (IV, 3), where the formulae for determining the coefficients c_r must be altered correspondingly to the new Equation (V, 8):

$$P_{c(\lambda)} = \mathfrak{v}_{\mathrm{in}(\tau_0)} \exp\left[-\lambda\,\tau_0\right] \qquad (V, 12)$$
$$+ (\lambda + \tau_0^{-1}\varrho - i\,\Omega) \int_0^{\tau_0} \mathfrak{v}_{\mathrm{in}(\tau)} \exp\left[-\lambda\,\tau\right] dt,$$

whereas the derivative $h'_{(\lambda)}$ is the same as before [apart from dividing it by m_{mech}, see (V, 9)]. Since all terms in the series (IV, 3), which contain characteristic roots (λ_r) with negative real part (a_r), are dying out for $t \to \infty$, the only surviving term is c_0, which belongs to a root with vanishing real part $(a_0 = 0)$. We know already that such a term occurs if Ω assumes one of the above values $\{\Omega_n\}$, and then the frequency b_0 of the final stationary motion becomes identical to Ω_n. Hence, the asymptotic form of the solutions is

$$\lim_{t\to\infty} \mathfrak{v}_{(t)} = (1+\varrho)^{-1}\left[\mathfrak{v}_{\mathrm{in}(\tau_0)} + \tau_0^{-1}\varrho\right. \qquad (V, 13)$$
$$\left. \cdot \int_0^{\tau_0} \mathfrak{v}_{\mathrm{in}(\tau)} \exp\left[-i\,\Omega_n\,\tau\right] d\tau\right] \cdot \exp\left[i\,\Omega_n t\right]$$

From this formula one recognizes quite clearly that the final state of motion is determined from the velocity $\mathfrak{v}_{\mathrm{in}(\tau_0)}$ at the end of the initial period and from the Fourier component of $\mathfrak{v}_{\mathrm{in}(\tau)}$ at the cyclotron frequency Ω_n. So we see that the external force oscillating with frequency Ω_n selects the internal mode of same frequency, whereas all other "non-resonant" modes are dying out (Figure 9).

However, we know already from the previous sections that for a general structure function $f_{(\sigma)}$ from (II, 9) such a resonance phenomenon is not possible. We conclude that such a particle exhibits a certain rigidity (in the relativistic sense) and resembles therefore a sort of "point particle", if one does not associate with this notion the vanishing size but rather thinks of the impossibility to excite interior vibrations (cf. the quite similar findings of Moniz and Sharp for their "point particle limit" [1]).

Let us now specialize the initial condition as $\mathfrak{v}_{\mathrm{in}(t)} = \mathfrak{v}_0 = \mathrm{const}$, which we had already used for our numerical calculations in Section III. For this initial condition, one concludes from (V, 13)
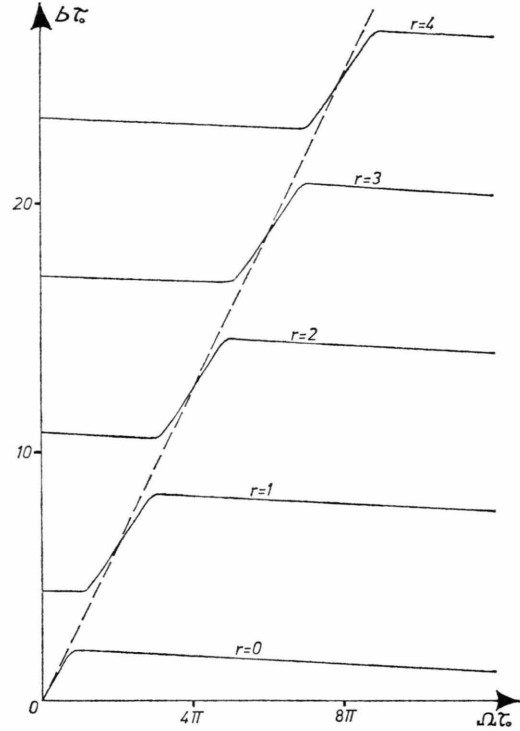

Fig. 9. The lowest positive frequencies b_r ($r = 0, 1, 2, 3, 4$) as function of Ω. Whenever Ω equals the frequency of an internal mode (dashed line: $b\,\tau_0 = \Omega\,\tau_0$), this mode becomes undamped. The frequencies b_r are almost constant with exception of the resonance region, where a characteristic transition from one constant value to the other one is performed.

$$\lim \mathfrak{v}_{(t)} = \mathfrak{v}_0(1+\varrho)^{-1} \exp\left[i\,\Omega_n t\right]$$
$$= \frac{m_{\mathrm{mech}}}{m_{\mathrm{exp}}} \mathfrak{v}_0 \exp\left[i\,\Omega_n t\right]. \qquad (V, 14)$$

Defining the absolute value of the non-relativistic total momentum by

$$P_{(t)} := m_{\mathrm{exp}}\left|\mathfrak{v}_{(t)}\right| \qquad (V, 15)$$

and its final value by P_∞, one verifies from (V, 14)

$$P_\infty = m_{\mathrm{mech}}\left|\mathfrak{v}_0\right| = P_{\mathrm{mech, in}}, \qquad (V, 16)$$

so that the irradiated part becomes $P_{\mathrm{rad}} := m_{\mathrm{el}}\left|\mathfrak{v}_0\right|$ and therefore

$$P_\infty + P_{\mathrm{rad}} = P_{\mathrm{in}} = m_{\mathrm{exp}}\left|\mathfrak{v}_0\right| = P_{\mathrm{mech, in}} + P_{\mathrm{el, in}}. \qquad (V, 17)$$

This confirms the numerical result of Sect. III, that only that part of the energy dragged along initially by the electromagnetic mass (m_{el}) alone, is radiated away; whereas the mechanical part is stored in the final motion. This statement can be generalized to an

arbitrary initial condition and one finds from (V, 13)

$$\mathfrak{P}_{\infty(t)} := m_{\exp} \mathfrak{v}_{\infty(t)} = \left\{ \mathfrak{P}_{\mathrm{mech,\,in}(\tau_0)} + \int_0^{\tau_0} \mathfrak{P}_{\mathrm{mech,\,in}(\tau)} \right.$$
$$\left. \exp\left[-i\,\Omega_n\,\tau \right] \frac{\mathrm{d}\tau}{\tau_0} \right\} \cdot \exp\left[i\,\Omega_n\,t \right] . \quad (V, 18)$$

This means that in addition to $\mathfrak{P}_{\mathrm{mech,\,in}(\tau_0)}$ the Fourier component of $\mathfrak{P}_{\mathrm{mech,\,in}(\tau)}$ at the resonance frequency Ω_n is preserved, whereas the energy contained in $\mathfrak{P}_{\mathrm{el}(\tau_0)}$ and in the non-resonant modes is radiated away.

The final problem to be solved in this section is the determination of the stationary radius (A). Contrary to the relativistic case, where the radius of the final circular motion is available only numerically, we can integrate here Eq. (V, 14) and find

$$\lim_{t\to\infty} \mathfrak{z}_{(t)} := \lim_{t\to\infty} \int^t \mathfrak{v}_{(t')}\,\mathrm{d}t' = \mathfrak{v}_0 [i\,\Omega_n(1+\varrho)]^{-1}$$
$$\cdot \exp[i\,\Omega_n\,t] . \quad (V, 19)$$

Hence, the stationary radius (A) is

$$A_n = |\mathfrak{v}_0| (\Omega_n[1+\varrho])^{-1} = (2\,\pi\,n\,[1+\varrho])^{-1}\,\tau_0 |\mathfrak{v}_0| . \quad (V, 20)$$

The linear dependance of A_1 from the initial velocity $|\mathfrak{v}_0| [= u(1+u^2)^{-1/2}]$ has also been entered in Figure 6. It becomes clear from this figure that the non-relativistic result (V, 20) is not too bad in comparison to the relativistic calculations (apart from a certain resonance character of the latter one), so that one can use (V, 20) for the following evaluations:

The maximal energy stored in the final circular motion is bounded for $u \to \infty$ ($\Rightarrow |\mathfrak{v}_0| \to c$) according to (III, 8) and (V, 20)

$$\mathfrak{E}_{\mathrm{fin}} = m_{\mathrm{mech}}\,c^2 \sqrt{1 + (2\,\pi\,n\,A_n\,\Delta s^{-1})^2}$$
$$\approx m_{\mathrm{mech}}\,c^2 \sqrt{1 + (1+\varrho)^{-2}\,\frac{|\mathfrak{v}_0|^2}{c^2}} ,$$
$$\mathfrak{E}_{\mathrm{fin,\,max}} \approx m_{\mathrm{mech}}\,c^2 \sqrt{1 + (1+\varrho)^{-2}} \leqq \sqrt{2}\,m_{\exp}\,c^2 . \quad (V, 21)$$

Hence, there can be stored up to 40% of the rest energy of the particle in the finale stationary state.

VI. The Longitudinal Motion

So far, we have restricted our considerations to the pure transverse motion, which is limited to a plane orthogonal to the magnetic lines of force. In this context, one is interested in the question, how an initial additional component of the velocity parallel to the magnetic field develops in time for both

cases where ω is equal or unequal to one of the resonance frequencies $n\,2\,\pi\,\Delta s^{-1}$.

The analogous problem for the Lorentz-Dirac and for the Mo-Papas equation [16] has already been solved [9] and it is found in both theories that an initial longitudinal component of 3-velocity is a constant of the motion. The energy loss due to radiation is taken here completely from the transverse velocity. Under certain initial conditions, the same result holds also for the general class (II, 1), as can be seen easily from the zero and third component of the equation of motion:

$$m_{\mathrm{mech}}\,c^2\,\dot{u}^3 + m_{\mathrm{el}}\,c^2 \{ \dot{Q}^0 - (\dot{Q}\,u)\,u^0 \} = 0 , \quad (VI, 1a)$$
$$m_{\mathrm{mech}}\,c^2\,\dot{u}^0 + m_{\mathrm{el}}\,c^2 \{ \dot{Q}^3 - (\dot{Q}\,u)\,u^3 \} = 0 . \quad (VI, 1b)$$

Putting here

$$u^3_{(s)} = c^{-1}\,v^3\,u^0_{(s)} ; \quad v^3 = \mathrm{const} \quad (VI, 2)$$

reduces the second Eq. (VI, 1b) to the first one (VI, 1a). Hence, if the longitudinal component v^3 is constant in the first Δs-interval, then it will remain constant throughout the motion.

However, if there is a non-vanishing acceleration in the first Δs-interval, then $v^3_{(s)}$ will not be constant. For numerical calculations in the relativistic case (I, 1), we have chosen $\mathrm{d}v^3_{(s)}/\mathrm{d}s = \dot{v}^3_{\mathrm{in}} = \mathrm{const}$ and the initial condition for the transverse component $v_{\perp(s)}$ was chosen to be constant: $v_{\perp(s)} = \mathrm{const}$. We have always observed the same qualitative behaviour of the longitudinal component $v^3_{(s)}$, no matter whether there existed resonance or not. Moreover, no serious difference was observed between the relativistic case and the non-relativistic limit. Figure 10 shows a typical result: within a proper
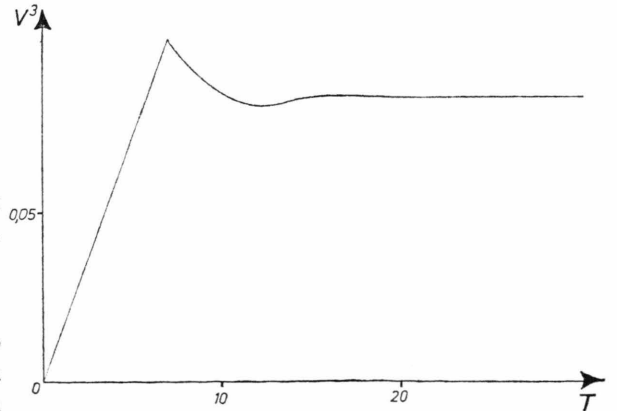

Fig. 10. Ordinary velocity v^3 versus laboratory time $T := c\,t/\Delta s_{\exp}$. After a characteristic time period of some τ_0-intervals, the longitudinal velocity v^3 assumes a constant value.

time interval Δs the longitudinal component $v^3_{(s)}$ is damped down to a constant value v^3_∞.

In the non-relativistic limit, we can again calculate exactly the final value v^3_∞, because the equations $(V, 1)$ (for $f_{(\tau)} \equiv 1$) are decoupled and the 3-component reads

$$m_{\text{mech}} \frac{\mathrm{d}v^3_{(t)}}{\mathrm{d}t} + m_{\text{el}} \tau_0^{-1} (v^3_{(t)} - v^3_{(t-\tau_0)}) = 0. \quad (VI, 3)$$

Hence, we can apply formulae $(IV, 3)$ to $(IV, 6)$ with the characteristic root λ_0 being equal to zero

(pure translation in the final state):

$$v^3_\infty \equiv c_0 = \dot{v}^3_{\text{in}} \tau_0 (1 - \tfrac{1}{2} m_{\text{el}}/m_{\text{exp}}). \quad (VI, 4)$$

So we see that the longitudinal velocity is also a constant of the motion, if one regards time intervals much larger than τ_0 (see Figure 10).

Acknowledgement

[1] E. J. Moniz and D. H. Sharp, Phys. Rev. **D 10,** 1133 [1974] and Preprint: "Radiation Reaction in Nonrelativistic Quantum Electrodynamics" [1977].

[2] D. I. Blokhintsev, "Space and Time in the Microworld", D. Reidel Publ. Co., Dortrecht-Holl., 1973, p. 95 ff.

[3] P. Caldirola, "Introduction of the Chronon in the Relativistic Theory of the Electron", Preprint 1977.

[4] F. Casagrande, "The Introduction of the Chronon into the Classical and Quantum Theory of the Electron", Preprint 1977.

[5] J. Petzold and M. Sorg, "Generalized Caldirola Equations for the Classically Radiating Electron", Submitted for publication in Z. Physik.

[6] M. Sorg, Z. Naturforsch. **32 a,** 659 [1977].

[7] W. Heudorfer and M. Sorg, Z. Naturforsch. **32 a,** 685 [1977].

[8] T. Erber, Fortschr. Phys. **9,** 343 [1961].

[9] H. Stöckel, Fortschr. Phys. **24,** 417 [1976].

[10] M. Sorg, Z. Naturforsch. **29 a,** 1671 [1974].

[11] J. C. Herrera, Phys. Rev. **D 15,** 453 [1977].

[12] N. D. Sen Gupta, Int. J. Theor. Phys. **4,** 389 [1971].

[13] D. White, Acta Phys. Austr. **45,** 331 [1976].

[14] K. Wildermuth, Z. Naturforsch. **10 a,** 450 [1955].

[15] R. Peierls, in "The Physicist's Conception of Nature", Reidel Publ. Co., Dortrecht-Holl., 1973, p. 374.

[16] N. D. Sen Gupta, Phys. Rev. **D 5,** 1546 [1972].

[17] R. Bellman and K. L. Cooke, "Differential-Difference Equations", Academic Press, 1963; Ch. 4.

[18] J. L. Kaplan, M. Sorg, and A. Yorke, Submitted for publication in Nonlinear Analysis.